Notes on Adjoint Operator

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The following two sections describe the derivations of the adjoint operators that are given in the paper "Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups" by Francesco Bullo, Naomi E. Leonard, and Andrew D. Lewis (IEEE Transactions on Automatic Control 45(8), pages 1437-1454, 2000). Specifically equation (1) on page 1438.

We know that,

$$ad_\xi \eta = [\widehat{\xi}, \widehat{\eta}],$$
$$[\widehat{\xi}, \widehat{\eta}] = \widehat{\xi}\widehat{\eta} - \widehat{\eta}\widehat{\xi}.$$

1 Adjoint operator on so(3)

Let $\xi = [\xi_1 \ \xi_2 \ \xi_3]^T$ and $\eta = [\eta_1 \ \eta_2 \ \eta_3]^T$. Then the corresponding elements in **so(3)** are

$$\widehat{\xi} = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix},$$

$$\widehat{\eta} = \begin{pmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix}.$$

Note that $\xi \times \eta = \widehat{\xi}\eta$.

We want to find the linear operator ad_ξ : **so(3)** $\rightarrow$ **so(3)**. Note that the result of the Lie bracket operation $[\widehat{\xi}, \widehat{\eta}]$ is a (3x3) skew-symmetric matrix which is an element of **so(3)**. There is a corresponding vector for this in $\Re^3$.

After performing the matrix operations $(\widehat{\xi}\widehat{\eta} - \widehat{\eta}\widehat{\xi})$, we get

$$[\widehat{\xi}, \widehat{\eta}] = \begin{pmatrix} 0 & \xi_2\eta_1 - \xi_1\eta_2 & \xi_3\eta_1 - \xi_1\eta_3 \\ -\xi_2\eta_1 + \xi_1\eta_2 & 0 & \xi_3\eta_2 - \xi_2\eta_3 \\ -\xi_3\eta_1 + \xi_1\eta_3 & -\xi_3\eta_2 + \xi_2\eta_3 & 0 \end{pmatrix}$$

The corresponding vector to this matrix in $\Re^3$ is $[(-\xi_3\eta_2 + \xi_2\eta_3) \ (\xi_3\eta_1 - \xi_1\eta_3) \ (-\xi_2\eta_1 + \xi_1\eta_2)]^T$ which is the cross product of ξ and η. Therefore

$$[\widehat{\xi}, \widehat{\eta}] = \widehat{\xi \times \eta}$$

Since $\xi \times \eta = \widehat{\xi}\eta$, we can write

$$\widehat{\xi \times \eta} = \widehat{(\widehat{\xi}\eta)}$$

Therefore, $ad_{\xi} = \widehat{\xi}$.

2 Adjoint operator on se(3)

Let $\xi, \eta \in \Re^6$ and $\widehat{\xi}, \widehat{\eta}, \in \mathbf{se(3)}$ where

$$\widehat{\xi} = \begin{pmatrix} \widehat{\Omega} & V \\ 0 & 0 \end{pmatrix},$$

$$\widehat{\eta} = \begin{pmatrix} \widehat{\Theta} & U \\ 0 & 0 \end{pmatrix}.$$

Note that $\widehat{\Omega}, \widehat{\Theta} \in \mathbf{so(3)}$ and $V, U \in \Re^3$. We want to find the linear operator $ad_{\xi} : \mathbf{se(3)} \to \mathbf{se(3)}$. After performing the matrix operations $(\widehat{\xi}\widehat{\eta} - \widehat{\eta}\widehat{\xi})$, we get

$$[\widehat{\xi}, \widehat{\eta}] = \begin{pmatrix} \widehat{\Omega}\widehat{\Theta} - \widehat{\Theta}\widehat{\Omega} & \widehat{\Omega}U - \widehat{\Theta}V \\ 0 & 0 \end{pmatrix}.$$

From previous section, we know that we can write (1,1) term of this matrix as follows

$$\widehat{\Omega}\widehat{\Theta} - \widehat{\Theta}\widehat{\Omega} = \widehat{\Omega \times \Theta} = \widehat{(\widehat{\Omega}\Theta)}.$$

Let's rewrite the second term (1,2) of $[\widehat{\xi}, \widehat{\eta}]$, using the property of the cross product as follows

$$\widehat{\Omega}U - \widehat{\Theta}V = \widehat{\Omega}U + \widehat{V}\Theta.$$

Note that $(\widehat{\Omega}U + \widehat{V}\Theta)$ is a vector in $\Re^3$.

In $\Re^6$, the element that corresponds to $[\widehat{\xi}, \widehat{\eta}]$ is

$$\begin{pmatrix} \widehat{\Omega}\Theta \\ \widehat{\Omega}U + \widehat{V}\Theta \end{pmatrix}.$$

We can write this vector as the multiplication of a vector and a linear map

$$\begin{pmatrix} \widehat{\Omega}\Theta \\ \widehat{\Omega}U + \widehat{V}\Theta \end{pmatrix} = \begin{pmatrix} \widehat{\Omega} & 0 \\ \widehat{V} & \widehat{\Omega} \end{pmatrix} \begin{pmatrix} \Theta \\ U \end{pmatrix}.$$

Therefore

$$ad_{\xi} = \begin{pmatrix} \widehat{\Omega} & 0 \\ \widehat{V} & \widehat{\Omega} \end{pmatrix}.$$